MATH 2028 - Proof of Change of Variables Theorem

GOAL: Give a proof of

Change of Variables Theorem

Let $g : A \to B$ be a diffeomorphism between two bounded open subsets $A, B \subseteq \mathbb{R}^n$ with measure zero boundary. For any cts $f : B \to \mathbb{R}$, we have

$$\boxed{\int_B f \, dV = \int_A (f \circ g) \cdot |\det(Dg)| \, dV} \qquad (*)$$

Proof: We will divide the proof into several steps.

Step 1: Suppose $g : U \to V$, $h : V \to W$ are diffeomorphisms between bdd open sets of $\mathbb{R}^n$ with measure zero boundary. If $(*)$ holds for both g and h, then $(*)$ also holds for $h \circ g$.

$$\int_W f \, dV = \int_V (f \circ h) \cdot |\det(Dh)| \, dV$$

$$= \int_U (f \circ h \circ g) \cdot |\det(Dh) \circ g| \cdot |\det(Dg)| \, dV$$

$\because$ Chain Rule $D(h\circ g) = (Dh\circ g)\cdot Dg$ $\displaystyle = \int_{U} (f\circ h\circ g)\cdot |\det D(h\circ g)|\, dV$

[**Step 2**: Suppose $\forall x \in A$, $\exists$ nbd. $U \subseteq A$ containing x s.t. $(*)$ holds for the restricted diffeomorphism $g|_U : U \to V := g(U)$ and all cts $f : V \to \mathbb{R}$ with cpt $\operatorname{spt}(f) \subseteq V$. THEN, $(*)$ holds for g.]

This is the step we need to use the partition of unity. Let U_i and $V_i = g(U_i)$ be the open sets in the assumption of Step 2. Note that

- $A = \bigcup_{i\in I} U_i$ and $B = \bigcup_{i \in I} V_i$

Choose a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ with cpt support w.r.t. the open cover $\{V_i\}_{i\in I}$ of B. Then

- $\{\varphi_\alpha \circ g\}_{\alpha\in A}$ is a partition of unity with cpt support w.r.t. the open cover $\{U_i\}_{i\in I}$ of A

[Exercise: Check this]

To check that (*) holds for g, let $f: B \to \mathbb{R}$ be a cts function. We proved in L8 that

$$\int_B f \, dV = \sum_{\alpha \in A} \int_B \varphi_\alpha \cdot f \, dV$$

For each $\alpha \in A$, $\exists\, i \in I$ s.t. $\mathrm{spt}(\varphi_\alpha) \subseteq V_i$.

Thus, $\displaystyle\int_B \varphi_\alpha \cdot f \, dV = \int_{V_i} \varphi_\alpha \cdot f \, dV$

By our hypothesis

$$\int_{V_i} \underbrace{\varphi_\alpha \cdot f}_{\mathrm{Spt} \subseteq V_i} \, dV = \int_{\mathcal{U}_i} \underbrace{(\varphi_\alpha \cdot f) \circ g}_{\mathrm{spt} \subseteq \mathcal{U}_i} \cdot |\det Dg| \, dV$$

$$\Rightarrow \int_B \varphi_\alpha \cdot f \, dV = \int_A (\varphi_\alpha \circ g) \cdot (f \circ g) \cdot |\det Dg| \, dV$$

Summing over $\alpha \in A$ on both sides, we used the theorem in L8 again to conclude that

$$\int_B f \, dV = \int_A (f \circ g) \cdot |\det Dg| \, dV$$

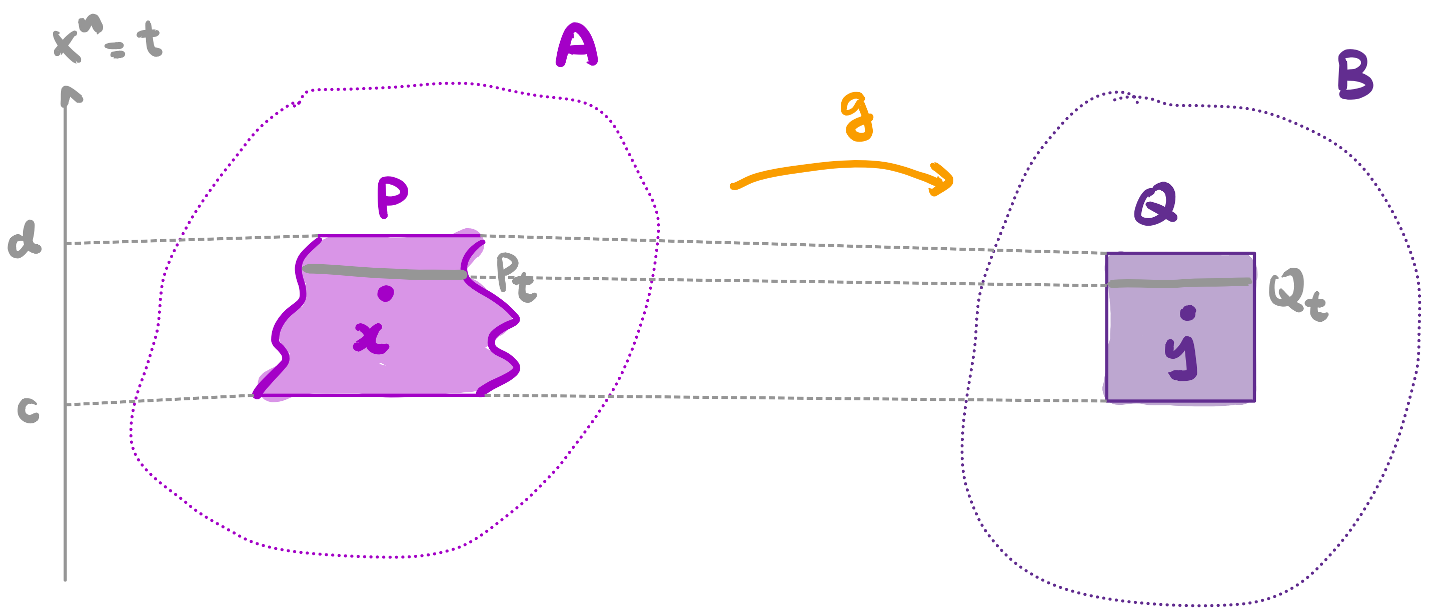
Step 3: $(*)$ holds if g is a diffeomorphism of the special form:

$$g(x_1,\dots,x_n) = (g_1(x_1,\dots,x_n),\dots,g_{n-1}(x_1,\dots,x_n),x_n)$$

We will argue by induction on n.

- $n=1$ is trivial as $g=\mathrm{id}$
- Suppose $(*)$ holds in dimension $n-1$.

Let $x \in A$ and $y = g(x) \in B$. Choose any rectangle $Q \subseteq B$ s.t. $y \in \operatorname{int} Q$ and denote $P = g^{-1}(Q) \subseteq A$ s.t. $x \in \operatorname{int} P$. Since g fixes the last coordinate x_n, we have:



By Step 2, it suffices to show that (*) holds for cts $f : \operatorname{int} Q \to \mathbb{R}$ with cpt spt $f \subseteq \operatorname{int} Q$

For each $t \in \mathbb{R}$, let $P_t = P \cap \{x_n = t\}$ and $Q_t = Q \cap \{x_n = t\}$. Note that

$\tilde{g} = g|_{P_t} : P_t \to Q_t$ is a diffeomorphism in $\mathbb{R}^{n-1}$

<u>Reason:</u> $\tilde{g}$ is clearly bijective and C^1.

$$Dg = \left(\begin{array}{c|c} D\tilde{g} & \begin{matrix} \frac{\partial g_1}{\partial x_n} \\ \vdots \\ \frac{\partial g_{n-1}}{\partial x_n} \end{matrix} \\ \hline 0 \cdots\cdots 0 & 1 \end{array}\right)$$

hence

$\det D\tilde{g} = \det Dg \neq 0$

THEN Inverse Function Theorem.

By induction hypothesis and Fubini's Thm.

$$\int_Q f \, dV = \int_c^d \int_{Q_t} f \, dx_1 \cdots dx_{n-1} \, dt$$

$$= \int_c^d \int_{P_t} (f \circ g) \cdot |\det Dg| \, dx_1 \cdots dx_{n-1} \, dt$$

$$= \int_P (f \circ g) \cdot |\det Dg| \, dV$$

Finally, combining Step 1 and Step 3, together with the following:

<u>FACT</u>: For any $a \in A$, $\exists$ nbd $\mathcal{U} \subseteq A$ of a s.t. $g|_{\mathcal{U}}$ can be decomposed into a composition of diffeomorphisms of the special form as in Step 3.

we are done proving the Change of Variables Theorem. $\square$